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Evaluating the Fractional Partial Derivatives of Some Two-Variables Fractional Functions

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Abstract: **In this paper, based on Jumarie's modified Riemann-Liouville (R-L) fractional derivative and a new multiplication of fractional analytic functions, we can find the fractional partial derivatives of some two-variables fractional functions. In fact, our result is a generalization of classical calculus result.**

Keywords: **Jumarie's modified R-L fractional derivative, new multiplication, fractional analytic functions, fractional partial derivatives, two-variables fractional functions.**

I. INTRODUCTION

Fractional calculus is a natural extension of the traditional calculus. In fact, since the beginning of the theory of differential and integral calculus, some mathematicians have studied their ideas on the calculation of non-integer order derivatives and integrals. During the 18th and 19th centuries, there were many famous scientists such as Euler, Laplace, Fourier, Abel, Liouville, Grunwald, Letnikov, Riemann, Laurent, Heaviside, and some others who reported interesting results within fractional calculus. With the development of computer technology, fractional calculus is widely used in various fields of science and engineering, such as physics, mechanics, electrical engineering, viscoelasticity, economics, bioengineering, and control theory [1-10].

However, the definition of fractional derivative is not unique. Commonly used definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, conformable fractional derivative, Jumarie's modified R-L fractional derivative [11-15]. Since Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with ordinary calculus.

In this paper, based on Jumarie type of Riemann-Liouville (R-L) fractional derivative and a new multiplication of fractional analytic functions, we can find the fractional partial derivatives of the following two types of two-variables fractional functions:

$$f_\alpha(x^\alpha, y^\alpha) = \left(s^2\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2} + t^2\left[\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha 2}\right)^{\otimes_\alpha\left(\frac{p}{2}\right)} \otimes_\alpha E_\alpha\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha\right] \otimes_\alpha cos_\alpha\left[t\frac{1}{\Gamma(\alpha+1)}y^\alpha + p\cdot arccot_\alpha\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha \otimes_\alpha \left[t\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha(-1)}\right]\right],$$

and

$$g_\alpha(x^\alpha, y^\alpha) = \left(s^2\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2} + t^2\left[\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha 2}\right)^{\otimes_\alpha\left(\frac{p}{2}\right)} \otimes_\alpha E_\alpha\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha\right] \otimes_\alpha sin_\alpha\left[t\frac{1}{\Gamma(\alpha+1)}y^\alpha + p\cdot arccot_\alpha\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha \otimes_\alpha \left[t\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha(-1)}\right]\right],$$

where $0 < \alpha \leq 1$, p is an integer, and s, t are real numbers. In fact, our result is a generalization of ordinary calculus result.

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II. PRELIMINARIES

Firstly, we introduce the fractional derivative used in this paper and its properties.

Definition 2.1 ([16])**:** Let $0 < \alpha \le 1$, and x_0 be a real number. The Jumarie type of Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\,. \tag{1}$$

where $\Gamma(\ )$ is the gamma function. On the other hand, for any positive integer m, we define $\left({}_{x_0}D_x^{\alpha}\right)^m[f(x)] = \left({}_{x_0}D_x^{\alpha}\right)\left({}_{x_0}D_x^{\alpha}\right)\cdots\left({}_{x_0}D_x^{\alpha}\right)[f(x)]$, the m-th order α-fractional derivative of $f(x)$.

Proposition 2.2 ([17])**:** *If α, β, x_0, C are real numbers and $\beta \ge \alpha > 0$, then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{2}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[C] = 0. \tag{3}$$

Next, we introduce the definition of fractional analytic function.

Definition 2.3 ([18])**:** If x, x_0, and a_k are real numbers for all k, $x_0 \in (a,b)$, and $0 < \alpha \le 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, i.e., $f_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. Furthermore, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

Next, a new multiplication of fractional analytic functions is introduced below.

Definition 2.4 ([19])**:** Let $0 < \alpha \le 1$, and x_0 be a real number. If $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}, \tag{4}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\,. \tag{5}$$

Then we define

$$\begin{aligned} & f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\otimes_\alpha\sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} \\ &= \sum_{n=0}^{\infty}\frac{1}{\Gamma(n\alpha+1)}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)(x-x_0)^{n\alpha}. \end{aligned} \tag{6}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}\otimes_\alpha\sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n} \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}. \end{aligned} \tag{7}$$

Definition 2.5 ([20])**:** If $0 < \alpha \le 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} = \sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}, \tag{8}$$

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$$g_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}=\sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes_\alpha n}. \tag{9}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha\circ g_\alpha)(x^\alpha)=f_\alpha(g_\alpha(x^\alpha))=\sum_{n=0}^{\infty}\frac{a_n}{n!}(g_\alpha(x^\alpha))^{\otimes_\alpha n}, \tag{10}$$

and

$$(g_\alpha\circ f_\alpha)(x^\alpha)=g_\alpha(f_\alpha(x^\alpha))=\sum_{n=0}^{\infty}\frac{b_n}{n!}(f_\alpha(x^\alpha))^{\otimes_\alpha n}. \tag{11}$$

Definition 2.6 ([21])**:** Let $0<\alpha\leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ be two α-fractional analytic functions. Then $(f_\alpha(x^\alpha))^{\otimes_\alpha n}=f_\alpha(x^\alpha)\otimes_\alpha\cdots\otimes_\alpha f_\alpha(x^\alpha)$ is called the nth power of $f_\alpha(x^\alpha)$. On the other hand, if $f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha)=1$, then $g_\alpha(x^\alpha)$ is called the $\otimes_\alpha$ reciprocal of $f_\alpha(x^\alpha)$, and is denoted by $(f_\alpha(x^\alpha))^{\otimes_\alpha -1}$.

Definition 2.7 ([22])**:** If $0<\alpha\leq 1$, and x is a real variable. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{x^{n\alpha}}{\Gamma(n\alpha+1)}=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha n}. \tag{12}$$

On the other hand, the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n\alpha}}{\Gamma(2n\alpha+1)}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2n}, \tag{13}$$

and

$$sin_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (2n+1)}. \tag{14}$$

Definition 2.8: If the complex number $z=p+iq$, where p,q are real numbers, and $i=\sqrt{-1}$. p, the real part of z, is denoted by $\mathrm{Re}(z)$; q the imaginary part of z, is denoted by $\mathrm{Im}(z)$.

Notation 2.9: If r is any real number, p is any positive integer. Define $(r)_p=r(r-1)\cdots(r-p+1)$, and $(r)_0=1$.

Definition 2.10: Let m,n be non-negative integers. For the two-variables fractional function $f_\alpha(x^\alpha,y^\alpha)$, its m-times fractional partial derivative with respect to x^α, n-times fractional partial derivative with respect to y^α, forms a $m+n$-th order fractional partial derivative, and denoted by $\left({}_{y_0}D_y^\alpha\right)^n\left({}_{x_0}D_x^\alpha\right)^m[f_\alpha(x^\alpha,y^\alpha)]$.

Proposition 2.11 (fractional Euler's formula)**:** *Let* $0<\alpha\leq 1$, *then*

$$E_\alpha(ix^\alpha)=cos_\alpha(x^\alpha)+isin_\alpha(x^\alpha). \tag{15}$$

Proposition 2.12 (fractional DeMoivre's formula)**:** *Let* $0<\alpha\leq 1$, *and* k *be a positive integer, then*

$$[cos_\alpha(x^\alpha)+isin_\alpha(x^\alpha)]^{\otimes_\alpha k}=cos_\alpha(kx^\alpha)+isin_\alpha(kx^\alpha). \tag{16}$$

III. MAIN RESULTS

In this section, we obtain fractional partial derivatives of two types of two-variables fractional functions. At first, a lemma is needed.

Lemma 3.1: *If* $0<\alpha\leq 1$, *and* p *is any integer,* x,y,s,t *are real numbers and* $t\frac{1}{\Gamma(\alpha+1)}y^\alpha\neq 0$. *Then*

$$\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha+it\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha p}$$

$$=\left(s^2\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}+t^2\left[\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha 2}\right)^{\otimes_\alpha\left(\frac{p}{2}\right)}\otimes_\alpha E_\alpha\left[ip\cdot arccot_\alpha\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha\otimes_\alpha\left[t\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha(-1)}\right]\right]. \tag{17}$$

Proof: $\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha + it\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha p}$

$$= \left[\left(s^2\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2} + t^2\left[\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha 2}\right)^{\otimes_\alpha\left(\frac{1}{2}\right)} \otimes_\alpha \left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha \otimes_\alpha \left(s^2\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2} + t^2\left[\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha 2}\right)^{\otimes_\alpha\left(-\frac{1}{2}\right)} + i\cdot t\frac{1}{\Gamma(\alpha+1)}y^\alpha \otimes_\alpha \left(s^2\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2} + t^2\left[\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha 2}\right)^{\otimes_\alpha\left(-\frac{1}{2}\right)}\right]\right]^{\otimes_\alpha p}$$

$$= \left(s^2\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2} + t^2\left[\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha 2}\right)^{\otimes_\alpha\left(\frac{p}{2}\right)} \otimes_\alpha [cos_\alpha(\theta^\alpha) + isin_\alpha(\theta^\alpha)]^{\otimes_\alpha p}$$

(where $\frac{1}{\Gamma(\alpha+1)}\theta^\alpha = arccot_\alpha\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha \otimes_\alpha \left[t\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha(-1)}\right]$)

$$= \left(s^2\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2} + t^2\left[\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha 2}\right)^{\otimes_\alpha\left(\frac{p}{2}\right)} \otimes_\alpha [cos_\alpha(p\theta^\alpha) + isin_\alpha(p\theta^\alpha)] \quad \text{(by fractional DeMoivre's formula)}$$

$$= \left(s^2\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2} + t^2\left[\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha 2}\right)^{\otimes_\alpha\left(\frac{p}{2}\right)} \otimes_\alpha \left[cos_\alpha\left(p\cdot arccot_\alpha\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha \otimes_\alpha \left[t\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha(-1)}\right]\right) + isin_\alpha\left(p\cdot arccot_\alpha\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha \otimes_\alpha \left[t\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha(-1)}\right]\right)\right]$$

$$= \left(s^2\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2} + t^2\left[\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha 2}\right)^{\otimes_\alpha\left(\frac{p}{2}\right)} \otimes_\alpha E_\alpha\left[ip\cdot arccot_\alpha\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha \otimes_\alpha \left[t\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha(-1)}\right]\right].$$

(by fractional Euler's formula) q.e.d.

Theorem 3.2: *Let* $0 < \alpha \le 1$, m, n *be non-negative integers,* p *be an integer, and* s, t *be real numbers, then the* $m + n$*-th order* α*-fractional partial derivative of the* α*-fractional function*

$$f_\alpha(x^\alpha, y^\alpha) = \left(s^2\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2} + t^2\left[\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha 2}\right)^{\otimes_\alpha\left(\frac{p}{2}\right)} \otimes_\alpha E_\alpha\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha\right] \otimes_\alpha cos_\alpha\left[t\frac{1}{\Gamma(\alpha+1)}y^\alpha + p\cdot arccot_\alpha\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha \otimes_\alpha \left[t\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha(-1)}\right]\right]$$ *is*

$$\left({}_{y_0}D_y^\alpha\right)^n\left({}_{x_0}D_x^\alpha\right)^m[f_\alpha(x^\alpha, y^\alpha)]$$

$$= s^m t^n cos\frac{m\pi}{2}\cdot\sum_{k=0}^\infty \frac{1}{k!}(k+p)_{m+n} cos_\alpha\left[(k+p-m-n)\cdot arccot_\alpha\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha \otimes_\alpha \left[t\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha(-1)}\right]\right]$$

$$-s^m t^n sin\frac{m\pi}{2}\cdot\sum_{k=0}^\infty \frac{1}{k!}(k+p)_{m+n} sin_\alpha\left[(k+p-m-n)\cdot arccot_\alpha\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha \otimes_\alpha \left[t\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha(-1)}\right]\right]. \quad (18)$$

Proof: By Lemma 3.1, we have

$$\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha + it\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha p} \otimes_\alpha E_\alpha\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha + it\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]$$

$$= \left(s^2\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}+t^2\left[\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha 2}\right)^{\otimes_\alpha\left(\frac{p}{2}\right)}\otimes_\alpha E_\alpha\left[ip\cdot arccot_\alpha\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha\otimes_\alpha\left[t\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha(-1)}\right]\right]$$

$$\otimes_\alpha E_\alpha\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha+it\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]$$

$$= \left(s^2\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}+t^2\left[\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha 2}\right)^{\otimes_\alpha\left(\frac{p}{2}\right)}\otimes_\alpha E_\alpha\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]\otimes_\alpha E_\alpha\left[i\left(t\frac{1}{\Gamma(\alpha+1)}y^\alpha+p\cdot\right.\right.$$

$$\left.\left. arccot_\alpha\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha\otimes_\alpha\left[t\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha(-1)}\right]\right)\right]. \quad (19)$$

Therefore,

$$f_\alpha(x^\alpha,y^\alpha)=\text{Re}\left\{\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha+it\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha p}\otimes_\alpha E_\alpha\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha+it\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]\right\}$$

$$=\text{Re}\left\{\sum_{k=0}^{\infty}\frac{1}{k!}\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha+it\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha(k+p)}\right\}. \quad (20)$$

Thus,

$$\left({}_{y_0}D_y^\alpha\right)^n\left({}_{x_0}D_x^\alpha\right)^m[f_\alpha(x^\alpha,y^\alpha)]$$

$$=\left({}_{y_0}D_y^\alpha\right)^n\left({}_{x_0}D_x^\alpha\right)^m\left[\text{Re}\left\{\sum_{k=0}^{\infty}\frac{1}{k!}\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha+it\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha(k+p)}\right\}\right]$$

$$=\text{Re}\left\{\sum_{k=0}^{\infty}\frac{1}{k!}\left({}_{y_0}D_y^\alpha\right)^n\left({}_{x_0}D_x^\alpha\right)^m\left[\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha+it\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha(k+p)}\right]\right\}$$

$$=s^mt^n\cdot\sum_{k=0}^{\infty}\frac{1}{k!}(k+p)_{m+n}\text{Re}\left\{i^n\left[\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha+it\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha(k+p-m-n)}\right]\right\}$$

$$=s^mt^n\cdot\sum_{k=0}^{\infty}\frac{1}{k!}(k+p)_{m+n}\text{Re}\left\{\left(cos\frac{m\pi}{2}+isin\frac{m\pi}{2}\right)\left[\left(s^2\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}+\right.\right.\right.$$

$$\left.\left.\left.t^2\left[\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha 2}\right)^{\otimes_\alpha\left(\frac{k+p-m-n}{2}\right)}\otimes_\alpha E_\alpha\left[i(k+p-m-n)\cdot arccot_\alpha\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha\otimes_\alpha\left[t\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha(-1)}\right]\right]\right]\right\}$$

$$=s^mt^ncos\frac{m\pi}{2}\cdot\sum_{k=0}^{\infty}\frac{1}{k!}(k+p)_{m+n}cos_\alpha\left[(k+p-m-n)\cdot arccot_\alpha\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha\otimes_\alpha\left[t\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha(-1)}\right]\right]$$

$$-s^mt^nsin\frac{m\pi}{2}\cdot\sum_{k=0}^{\infty}\frac{1}{k!}(k+p)_{m+n}sin_\alpha\left[(k+p-m-n)\cdot arccot_\alpha\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha\otimes_\alpha\left[t\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha(-1)}\right]\right].$$

q.e.d.

Theorem 3.3: *If the assumptions are the same as Theorem 3.1, then the* $m+n$*-th order* α*-fractional partial derivative of the* α*-fractional function*

$$g_\alpha(x^\alpha,y^\alpha)=\left(s^2\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}+t^2\left[\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha 2}\right)^{\otimes_\alpha\left(\frac{p}{2}\right)}\otimes_\alpha E_\alpha\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]\otimes_\alpha sin_\alpha\left[t\frac{1}{\Gamma(\alpha+1)}y^\alpha+p\cdot\right.$$

$$\left. arccot_\alpha\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha\otimes_\alpha\left[t\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha(-1)}\right]\right]$$ *is*

$$\left({}_{y_0}D_y^\alpha\right)^n\left({}_{x_0}D_x^\alpha\right)^m[g_\alpha(x^\alpha,y^\alpha)]$$

$$= s^m t^n \sin\frac{m\pi}{2} \cdot \sum_{k=0}^{\infty} \frac{1}{k!}(k+p)_{m+n} \cos_\alpha\left[(k+p-m-n)\cdot arccot_\alpha\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha \otimes_\alpha \left[t\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha(-1)}\right]\right]$$

$$+ s^m t^n \cos\frac{m\pi}{2} \cdot \sum_{k=0}^{\infty} \frac{1}{k!}(k+p)_{m+n} \sin_\alpha\left[(k+p-m-n)\cdot arccot_\alpha\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha \otimes_\alpha \left[t\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha(-1)}\right]\right]. \quad (21)$$

Proof : Since

$$g_\alpha(x^\alpha, y^\alpha) = \text{Im}\left\{\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha + it\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha p} \otimes_\alpha E_\alpha\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha + it\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]\right\}$$

$$= \text{Im}\left\{\sum_{k=0}^{\infty}\frac{1}{k!}\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha + it\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha(k+p)}\right\}. \quad (22)$$

It follows that

$$\left({}_{y_0}D_y^\alpha\right)^n \left({}_{x_0}D_x^\alpha\right)^m [g_\alpha(x^\alpha, y^\alpha)]$$

$$= \left({}_{y_0}D_y^\alpha\right)^n \left({}_{x_0}D_x^\alpha\right)^m \left[\text{Im}\left\{\sum_{k=0}^{\infty}\frac{1}{k!}\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha + it\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha(k+p)}\right\}\right]$$

$$= \text{Im}\left\{\sum_{k=0}^{\infty}\frac{1}{k!}\left({}_{y_0}D_y^\alpha\right)^n \left({}_{x_0}D_x^\alpha\right)^m \left[\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha + it\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha(k+p)}\right]\right\}$$

$$= s^m t^n \cdot \sum_{k=0}^{\infty}\frac{1}{k!}(k+p)_{m+n}\text{Im}\left\{i^n\left[\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha + it\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha(k+p-m-n)}\right]\right\}$$

$$= s^m t^n \cdot \sum_{k=0}^{\infty}\frac{1}{k!}(k+p)_{m+n}\text{Im}\left\{\left(\cos\frac{m\pi}{2} + i\sin\frac{m\pi}{2}\right)\left[\left(s^2\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2} + \right.\right.\right.$$

$$\left.\left.\left. t^2\left[\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha 2}\right)^{\otimes_\alpha\left(\frac{k+p-m-n}{2}\right)} \otimes_\alpha E_\alpha\left[i(k+p-m-n)\cdot arccot_\alpha\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha \otimes_\alpha \left[t\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha(-1)}\right]\right]\right]\right\}$$

$$= s^m t^n \sin\frac{m\pi}{2} \cdot \sum_{k=0}^{\infty} \frac{1}{k!}(k+p)_{m+n} \cos_\alpha\left[(k+p-m-n)\cdot arccot_\alpha\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha \otimes_\alpha \left[t\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha(-1)}\right]\right]$$

$$+ s^m t^n \cos\frac{m\pi}{2} \cdot \sum_{k=0}^{\infty} \frac{1}{k!}(k+p)_{m+n} \sin_\alpha\left[(k+p-m-n)\cdot arccot_\alpha\left[s\frac{1}{\Gamma(\alpha+1)}x^\alpha \otimes_\alpha \left[t\frac{1}{\Gamma(\alpha+1)}y^\alpha\right]^{\otimes_\alpha(-1)}\right]\right].$$

q.e.d.

IV. CONCLUSION

In this paper, the fractional partial derivatives of two types of two-variables fractional functions are obtained based Jumarie's modified R-L fractional derivative and a new multiplication of fractional analytic functions. In fact, our result is a generalization of traditional calculus result. In the future, we will continue to study the problems in applied mathematics and fractional differential equations.

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